

# Spanning trails with maximum degree at most 4 in $2K_2$ -free graphs

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**Abstract.** A graph is called  $2K_2$ -free if it does not contain two independent edges as an induced subgraph. Mou and Pasechnik conjectured that every  $\frac{3}{2}$ -tough  $2K_2$ -free graph with at least three vertices has a spanning trail with maximum degree at most 4. In this paper, we confirm this conjecture. We also provide examples for all  $t < \frac{5}{4}$  of  $t$ -tough graphs that do not have a spanning trail with maximum degree at most 4.

**Keywords.** Toughness;  $2K_2$ -free graph; 2-trail; dominating cycle

## 1 Introduction

Graphs considered in this paper are simple, undirected, and finite. Let  $G$  be a graph. Let  $V(G)$  and  $E(G)$  be the vertex set and edge set of  $G$ , respectively. For  $v \in V(G)$ ,  $N_G(v)$  denotes the set of neighbors of  $v$  in  $G$ , and  $d_G(v) = |N_G(v)|$  the degree of  $v$  in  $G$ . If  $S \subseteq V(G)$  then the subgraph induced by  $V(G) - S$  is denoted by  $G - S$ . For notational simplicity we write  $G - \{x\}$  for  $G - x$ . Let  $u, v \in V(G)$  be two vertices. Then  $\text{dist}_G(u, v)$ , the *distance between  $u$  and  $v$  in  $G$* , is defined to be the length of a shortest path connecting  $u$  and  $v$  in  $G$ . If  $uv \notin E(G)$ , we write  $G + uv$  for the new graph obtained from  $G$  by adding the edge  $uv$ . If  $uv \in E(G)$ , then  $G - uv$  denotes the graph obtained from  $G$  by deleting the edge  $uv$ . Let  $V_1, V_2 \subseteq V(G)$  be two disjoint sets. Then  $E_G(V_1, V_2)$  is the set of edges of  $G$  with one end in  $V_1$  and the other end in  $V_2$ . The graph  $G$  is called  $2K_2$ -free if it does not contain two independent edges as an induced subgraph.

The number of components of  $G$  is denoted by  $c(G)$ . Let  $t \geq 0$  be a real number. The

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graph is said to be  $t$ -tough if  $|S| \geq t \cdot c(G - S)$  for each  $S \subseteq V(G)$  with  $c(G - S) \geq 2$ . The *toughness*  $\tau(G)$  is the largest real number  $t$  for which  $G$  is  $t$ -tough, or is defined as  $\infty$  if  $G$  is complete. This concept, a measure of graph connectivity and “resilience” under removal of vertices, was introduced by Chvátal [5] in 1973. It is easy to see that if  $G$  has a hamiltonian cycle then  $G$  is 1-tough. Conversely, Chvátal [5] conjectured that there exists a constant  $t_0$  such that every  $t_0$ -tough graph is hamiltonian. Bauer, Broersma and Veldman [1] have constructed  $t$ -tough graphs that are not hamiltonian for all  $t < \frac{9}{4}$ , so  $t_0$  must be at least  $\frac{9}{4}$ .

There are a number of papers on Chvátal’s toughness conjecture, and it has been verified when restricted to a number of graph classes [2], including planar graphs, claw-free graphs, co-comparability graphs, and chordal graphs. Recently, Broersma, Patel and Pyatkin [3] proved that every 25-tough  $2K_2$ -free graph on at least three vertices is hamiltonian.

Another direction inspired by Chvátal’s toughness conjecture is investigating the existence of spanning substructures weaker than hamiltonian cycles for a given toughness. For example,  $k$ -trees,  $k$ -walks, and  $k$ -trails are substructures of this kind. Let  $k$  be a positive integer. A  $k$ -tree is a tree with maximum degree at most  $k$ , and a  $k$ -walk is a closed walk with each vertex repeated at most  $k$  times. A  $k$ -walk can be obtained from a  $k$ -tree by visiting each edge of the tree twice. A  $k$ -trail is a  $k$ -walk with no repetition of edges. A graph has a spanning  $k$ -trail if and only if it has a spanning Eulerian subgraph with maximum degree at most  $2k$ . A spanning 2-tree is just a hamiltonian path and a spanning 1-walk/1-trail is a hamiltonian cycle.

In 1990, Jackson and Wormald [8] made the following conjecture.

**Conjecture 1.** *Let  $k \geq 2$  be a positive integer. Then every  $\frac{1}{k-1}$ -tough graph has a spanning  $k$ -walk.*

Mou and Pasechnik [12, 11] confirmed Jackson and Wormald’s conjecture for  $2K_2$ -free graphs. In [11], they proposed the following two conjectures.

**Conjecture 2.** *Every  $\frac{3}{2}$ -tough  $2K_2$ -free graph with at least three vertices has a spanning 2-trail.*

**Conjecture 3.** *Every 2-tough  $2K_2$ -free graph with at least three vertices is hamiltonian.*

The class of  $2K_2$ -free graphs is well studied, for instance, see [3, 4, 6, 10, 12, 11, 13]. It is a superclass of *split* graphs, where the vertices can be partitioned into a clique and an independent set. One can also easily check that every *cochordal* graph (i.e., a graph that is the complement of a chordal graph) is  $2K_2$ -free and so the class of  $2K_2$ -free graphs is at least as rich as the class of chordal graphs.

In this paper, we confirm Conjecture 2.

**Theorem 1.** *Let  $G$  be a  $\frac{3}{2}$ -tough  $2K_2$ -free graph with at least three vertices. Then  $G$  has a spanning 2-trail.*

There is a large literature proving the existence of a spanning closed trail under various conditions; a graph with a spanning closed trail is called *supereulerian*. A recent paper in this area, providing references to other papers, is [9]. However, apart from results on hamiltonicity there do not seem to be many results on spanning closed trails with bounded degree. Other than Theorem 1, the only one we are aware of is in [7], which proves that a 2-edge-connected  $n$ -vertex graph  $G$  with  $n \geq 7$  and  $\sigma_3(G) \geq n$  has a spanning 2-trail, where  $\sigma_3(G)$  is the minimum degree sum over all triples of pairwise independent vertices.

We prove Theorem 1 in Section 2. In Section 3, we construct  $2K_2$ -free graphs with toughness close to  $\frac{5}{4}$  but containing no spanning 2-trail.

## 2 Proof of Theorem 1

We need the following lemma in proving Theorem 1.

**Lemma 2.1.** *Let  $G$  be a bipartite graph with partite sets  $X$  and  $Y$ . If for every  $S \subseteq X$ ,  $|N_G(S)| \geq \frac{3}{2}|S|$ , then  $G$  has a subgraph  $H$  covering  $X$  (meaning that  $X \subseteq V(H)$ ) such that for every  $x \in X$ ,  $d_H(x) = 2$  and for every  $y \in Y$ ,  $d_H(y) \leq 2$ .*

**Proof.** Form  $G'$  from  $G$  by replacing each  $x \in X$  by  $x_1, x_2, x_3$ , each  $y \in Y$  by  $y_1, y_2$ , and each  $xy \in E(G)$  by six edges  $x_i y_j$ ,  $1 \leq i \leq 3$ ,  $1 \leq j \leq 2$ . Let  $\pi$  be the natural projection from  $G'$  to  $G$  with

$$\pi(x_i) = x, \quad \pi(y_j) = y, \quad \pi(x_i y_j) = xy.$$

Let  $X' = \pi^{-1}(X)$  be the inverse image of  $X$  under  $\pi$ . For each  $S' \subseteq X'$ , let  $S = \pi(S')$ . Then  $|S'| \leq 3|S| \leq 2|N_G(S)| = |N_{G'}(S')|$ . Thus, by Hall's Theorem,  $G'$  has a matching  $M'$  covering  $X'$ . The projection  $\pi(M')$  of  $M'$  is a graph containing all the vertices in  $X$  such that each vertex in  $X$  has degree 2 or 3, and each vertex in  $Y$  has degree at most 2. In  $\pi(M')$ , for each  $x \in X$  with degree 3, delete one edge incident to  $x$ . Then the graph  $H$  induced by the remaining edges is the desired graph.  $\square$

We cannot reduce the number  $\frac{3}{2}$  in Lemma 2.1. To see this, take  $k \geq 1$ ,  $X$  with  $|X| = 2k$ , and  $Y = Y_1 \cup Y_2$  with  $|Y_1| = 2k$  and  $|Y_2| = k$ . To form  $G$ , join each vertex of  $X$  to a distinct vertex of  $Y_1$  (giving a matching) and join every vertex of  $X$  to every vertex of  $Y_2$ . Then  $G$  has a subgraph  $H$  as described, but if we delete any  $y \in Y$  then no such subgraph exists although  $G - y$  satisfies the condition of Lemma 2.1 with  $\frac{3k-1}{2k}$  instead of  $\frac{3}{2}$ .

A subgraph  $G^* \subseteq G$  is called *dominating* if  $G - V(G^*)$  is an edgeless graph. Mou and Pasechnik proved the existence of a dominating cycle in  $2K_2$ -free graphs. In fact, the proof

of [12, Theorem 3] implies the following.

**Lemma 2.2.** *Let  $G$  be a  $2K_2$ -free graph containing a cycle. Then some longest cycle of  $G$  is dominating.*

**Proof of Theorem 1.** As  $G$  is  $\frac{3}{2}$ -tough,  $G$  is 3-connected. So  $G$  has a cycle. Let  $C$  be a dominating longest cycle of  $G$ , which exists by Lemma 2.2. Let  $\vec{C}$  denote a forward orientation of  $C$ . For a vertex  $x \in V(C)$ , we let  $x^+$  denote the successor of  $x$  on  $\vec{C}$ , and if  $S \subseteq V(C)$  we define  $S^+ = \{x^+ \mid x \in S\}$ . We may assume  $V(G) - V(C) \neq \emptyset$ . Otherwise,  $C$  is a spanning 1-trail.

**Claim A:** Let  $x \in V(G) - V(C)$ .

- (a)  $N_G(x)$  does not contain two consecutive vertices on  $C$ .
- (b) If  $y, z \in N_G(x)$  with  $y \neq z$  then there is no path from  $y^+$  to  $z^+$  that is internally disjoint from  $C$ ; in particular,  $y^+z^+ \notin E(G)$ .
- (c)  $C$  has at least 7 vertices.

**Proof.** Both (a) and (b) follow by standard arguments. We only prove (c) here. Since  $G - V(C)$  is edgeless,  $N_G(x) \subseteq V(C)$ . By (a),  $N_G(x)^+$  is disjoint from  $N_G(x)$ . As  $G$  is  $\frac{3}{2}$ -tough,  $\delta(G) \geq 3$ , so  $|N_G(x)| = |N_G(x)^+| \geq 3$ . Thus,  $|V(C)| \geq 6$ , and  $|V(C)| = 6$  precisely when  $|N_G(x)| = 3$  and  $V(C) = N_G(x) \cup N_G(x)^+$ . In that case, by (b) the vertices of  $N_G(x)^+$  belong to separate components in  $G - N_G(x)$ . Thus,  $c(G - N_G(x)) \geq 4$ , and so  $\frac{|N_G(x)|}{c(G - N_G(x))} \leq \frac{3}{4} < \frac{3}{2}$ , contradicting the toughness of  $G$ .  $\square$

Let  $G' = G - E(G[V(C)])$  with partite sets  $X = V(G) - V(C)$  and  $Y = V(C)$ . Since  $G$  is  $\frac{3}{2}$ -tough and  $X$  is an independent set in  $G$ , we have that for any  $S \subseteq X$ ,  $|N_{G'}(S)| \geq \frac{3}{2}|S|$  (even when  $|S| = 1$ , because then  $c(G - N_{G'}(S)) \geq 2$  by (a) of Claim A). Applying Lemma 2.1 to  $G'$ , we see that  $G'$  (hence  $G$ ) has a subgraph  $H$  such that for any  $x \in X$ ,  $d_H(x) = 2$  and for any  $y \in Y \cap V(H)$ ,  $d_H(y) = 1$  or  $d_H(y) = 2$ . Subject to this property, we choose a subgraph  $H$  of  $G$  such that the number of components in  $H$  is smallest. Let  $H_1, \dots, H_\ell$  be the components of  $H$ . Each  $H_i$  is either a path or a cycle. Assume, without loss of generality, that  $H_1, \dots, H_p$  are paths and  $H_{p+1}, \dots, H_\ell$  are cycles. For each path  $H_i$  ( $1 \leq i \leq p$ ), let  $u_i$  and  $v_i$  denote its endvertices (these two vertices are on  $C$  by the construction of  $H$ ). Let  $s_i$  and  $t_i$  denote the neighbor of  $u_i$  and  $v_i$  in  $H$ , respectively. Note that  $s_i$  and  $t_i$  are vertices from  $V(G) - V(C)$  and  $s_i = t_i$  if  $H_i$  has length 2. Note also that  $C \cup \left(\bigcup_{p+1 \leq i \leq \ell} H_i\right)$  is a spanning 2-trail if  $p = 0$ . Therefore, we assume  $p \geq 1$ .

**Claim B:** Each of the following holds.

- (a)  $s_i u_j, s_i v_j, t_i u_j, t_i v_j \notin E(G)$ , for all  $i, j$  with  $i \neq j$  and  $i, j \in \{1, \dots, p\}$ .
- (b) Let  $u$  be an endvertex of  $H_i$  and  $v$  be an endvertex of  $H_j$ , where  $i \neq j$  and  $i, j \in \{1, \dots, p\}$ . Then  $uv \in E(G)$ .

**Proof.** For (a), if say  $s_i u_j \in E(G)$  then we could replace  $s_i u_i$  by  $s_i u_j$  in  $H$  to obtain fewer components. For (b), let  $s$  be the neighbor of  $u$  on  $H_i$ , and  $t$  be the neighbor of  $v$  on  $H_j$ . Note that  $s, t \in V(G) - V(C)$ . Since  $i \neq j$ , we have  $s \neq t$ . By (a), we have  $sv, tu \notin E(G)$ . Furthermore,  $st \notin E(G)$  as  $G - V(C)$  is edgeless. So  $uv \in E(G)$  by the  $2K_2$ -freeness of  $G$ .  $\square$

**Claim C:** Let  $q$  be an integer with  $1 \leq q \leq p$ , and let  $V_q = \bigcup_{1 \leq i \leq q} V(H_i)$ . Then  $G[V_q] - E(C)$  contains a path  $P_q$  with vertex set  $V_q$  such that for each  $i$  with  $1 \leq i \leq q$ ,  $H_i$  is a subpath of  $P_q$  and both endvertices of  $P_q$  belong to  $\{u_1, \dots, u_q, v_1, \dots, v_q\}$ .

**Proof.** We show this claim by induction on  $q$ . For  $q = 1$ ,  $H_1$  itself is a desired path. So we assume that  $q \geq 2$ . By the induction hypothesis,  $G[V_{q-1}] - E(C)$  contains a path  $P_{q-1}$  with the desired property. Assume, without loss of generality, that the two endvertices of  $P_{q-1}$  are  $u_a$  and  $v_b$  with  $a, b \in \{1, \dots, q-1\}$ . As  $|V(C)| \geq 7$  by (b) of Claim A, we see that one of  $\text{dist}_C(u_a, u_q), \text{dist}_C(u_a, v_q), \text{dist}_C(v_b, u_q), \text{dist}_C(v_b, v_q)$  must be at least 2. Assume, without loss of generality, that  $\text{dist}_C(u_a, v_q) \geq 2$ . Then  $u_a v_q \in E(G)$  by (b) of Claim B and  $u_a v_q \in E(G) - E(C)$  since  $\text{dist}_C(u_a, v_q) \geq 2$ . Thus,  $P_{q-1} \cup H_q + u_a v_q$  is a desired path.  $\square$

Let  $D = \bigcup_{p+1 \leq i \leq \ell} H_i$  be the union of the cycle components of  $H$ . Consider two cases.

**Case 1:**  $p \geq 2$ .

Let  $P_p$  be a path with the property stated in Claim C. Assume, without loss of generality, that the endvertices of  $P_p$  are  $u_1$  and  $v_p$ . By (b) of Claim B, we have  $v_p u_1 \in E(G)$ . Let

$$T = \begin{cases} C \cup D \cup P_p - v_p u_1, & \text{if } v_p u_1 \in E(C); \\ C \cup D \cup P_p + v_p u_1, & \text{if } v_p u_1 \in E(G) - E(C). \end{cases}$$

Then  $T$  is a spanning 2-trail of  $G$ .

**Case 2:**  $p = 1$ .

Assume first that  $|V(H_1)| \geq 4$ . Consider the two edges  $s_1 u_1$  and  $t_1 v_1$ . Again, we have  $\{s_1 v_1, t_1 u_1, u_1 v_1\} \cap E(G) \neq \emptyset$  by the  $2K_2$ -freeness of  $G$ . Let

$$T = \begin{cases} C \cup H - s_1 u_1 + s_1 v_1, & \text{if } s_1 v_1 \in E(G); \\ C \cup H - t_1 v_1 + t_1 u_1, & \text{if } t_1 u_1 \in E(G); \\ C \cup H - u_1 v_1, & \text{if } u_1 v_1 \in E(C); \\ C \cup H + u_1 v_1, & \text{if } u_1 v_1 \in E(G) - E(C). \end{cases}$$

Then  $T$  is a spanning 2-trail of  $G$ .

Assume now that  $|V(H_1)| = 3$ . Suppose that  $\text{dist}_C(u_1, v_1) \geq 3$ . As  $u_1^+ v_1^+ \notin E(G)$  by (b) of Claim A, we have  $\{u_1 v_1, u_1 v_1^+, v_1 u_1^+\} \cap E(G) \neq \emptyset$  by the  $2K_2$ -freeness of  $G$ . Note

that  $\{u_1v_1, u_1v_1^+, v_1u_1^+\} \cap E(C) = \emptyset$  as  $\text{dist}_C(u_1, v_1) \geq 3$ . Let

$$T = \begin{cases} C \cup H + u_1v_1^+ - v_1v_1^+, & \text{if } u_1v_1^+ \in E(G); \\ C \cup H + v_1u_1^+ - u_1u_1^+, & \text{if } v_1u_1^+ \in E(G); \\ C \cup H + u_1v_1, & \text{if } u_1v_1 \in E(G). \end{cases}$$

In the first case the vertex  $v_1^+$  may also be contained in  $D$ , but when we add the edge  $u_1v_1^+$  and remove the edge  $v_1v_1^+$ , the degree of  $v_1^+$  in  $T$  is the same as in  $C \cup H$ . The same applies to  $u_1^+$  in the second case. Thus the degree of each vertex in  $T$  is at most 4, and  $T$  is a spanning 2-trail of  $G$ .

Suppose that  $N_G(s_1) - V(H) \neq \emptyset$ . Then  $N_G(s_1) - V(D)$ , which includes  $u_1$  and  $v_1$ , contains at least three vertices. By Claim A, these vertices are pairwise nonadjacent and  $|V(C)| \geq 7$ , so there are  $u', v' \in N_G(s_1) - V(D)$  with  $\text{dist}_C(u', v') \geq 3$ . We replace  $H_1$  by the path  $u's_1v'$  and apply the argument above.

Therefore, we assume all neighbors of  $s_1$  not in  $H_1$  lie in  $D$ , which must be nonempty. Suppose that  $N_G(x') \subseteq V(H)$  for all  $x' \in X \cap V(D)$ . Then deleting all the  $|X| + 1$  neighbors of vertices in  $X$  on  $C$  results in at least  $|X|$  components. Since  $|V(D) \cap X| \geq 2$  and  $s_1 \in X$ ,  $|X| \geq 3$ , so  $\frac{|X|+1}{|X|} \leq \frac{4}{3} < \frac{3}{2}$ , contradicting the toughness of  $G$ . Therefore there exist  $x'$  and  $u'$  with  $x' \in X \cap V(D)$  and  $u' \in N_G(x') - V(H)$ . Let  $x'v' \in E(D)$  and  $D' = D - x'v' + x'u'$ . Replacing  $D$  by  $D'$  in  $H$ , we see that the new graph has the same property as  $H$ , but it has two components that are paths, so we may apply Case 1.

The proof of Theorem 1 is now complete.  $\square$

### 3 An Extremal Example

In this section, we construct a family of  $2K_2$ -free graphs with toughness approaching  $\frac{5}{4}$  that do not contain any spanning 2-trail.

Let  $n \geq 2$  be an integer,  $Q_1 = K_{4n}$ , the complete graph on  $4n$  vertices,  $Q_2 = \overline{K_{4n}}$ , the empty graph on  $4n$  vertices, and  $Q_3 = K_{n-1}$ . Let  $G_n$  be a graph with  $V(G_n) = V(Q_1) \cup V(Q_2) \cup V(Q_3)$  and  $E(G_n)$  consisting of all edges in  $Q_1$  and  $Q_3$ , all edges between  $V(Q_3)$  and  $V(Q_1) \cup V(Q_2)$ , and a perfect matching between  $Q_1$  and  $Q_2$ . It is easy to check that  $G$  is  $2K_2$ -free.

We claim that  $\lim_{n \rightarrow \infty} \tau(G_n) = \frac{5}{4}$ . Let  $S \subseteq V(G_n)$  be a cutset such that  $\tau(G_n) = \frac{|S|}{c(G_n - S)}$ . Then  $Q_3 \subseteq S$  as each vertex in  $Q_3$  is adjacent to every other vertex of  $G_n$ . Also,  $S \cap V(Q_2) = \emptyset$ . Otherwise, as  $c(G - (S - V(Q_2))) \geq c(G - S)$ , we get  $\frac{|S - V(Q_2)|}{c(G_n - (S - V(Q_2)))} < \frac{|S|}{c(G_n - S)} = \tau(G_n)$ , contradicting the toughness of  $G$ . Thus,  $c(G - S) = |S \cap V(Q_1)| + 1$  if  $V(Q_1) \not\subseteq S$  and  $c(G - S) = 4n$  otherwise. In the latter case,  $\frac{|S|}{c(G_n - S)} = \frac{5n-1}{4n}$ . So assume  $V(Q_1) \not\subseteq S$  and  $|V(Q_1) \cap S| = r$ , where  $1 \leq r \leq 4n - 1$ . Then  $\frac{n-1+r}{r+1}$  is a decreasing function of  $r$  which

achieves its minimum when  $r = 4n - 1$ . Hence,  $\tau(G_n) = \frac{|S|}{c(G_n - S)} = \frac{5n-2}{4n}$ , which approaches  $\frac{5}{4}$  as  $n \rightarrow \infty$ .

We show now that  $G_n$  has no spanning 2-trail. Suppose on the contrary that  $T$  is a spanning 2-trail of  $G_n$ . Let  $v \in V(Q_2)$  be a vertex. Then  $d_T(v) \geq 2$ . As  $|N_G(v) \cap V(Q_1)| = 1$ ,  $|N_T(v) \cap V(Q_3)| \geq 1$ . Thus,  $|E_T(V(Q_3), V(Q_2))| \geq 4n$ . Since  $|V(Q_3)| = n - 1$ , by the Pigeonhole Principle there is a vertex from  $Q_3$  that has degree at least 5 in  $T$ . This contradicts the assumption that  $T$  is a 2-trail.

From the example above, we suspect the following might be true.

**Conjecture 4.** *Any  $\frac{5}{4}$ -tough  $2K_2$ -free graph with at least three vertices has a spanning 2-trail.*

Our proof of Theorem 1 relies on Lemma 2.1, which cannot be improved, so a new strategy will be needed to obtain a positive answer to this conjecture.

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